ON THE PROBLEM OF PURSUIT IN THE CASE OF LINEAR MONOTYPE OBJECTS

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This paper is concerned with the problem [1] of the minimax time elapsing before the encounter of two linearly controlled objects described by identical equations. Furthermore, it is assumed that constraints on the controlling forces allow only continuous motions of the objects.

1. Let us consider the problem [1] on the minimax time T elapsing before the encounter of the pursuing $(y(t))$ and persued $(z(t))$ motions, which are described respectively by Equations

$$dy / dt = Ay + Bu, \qquad dz / dt = Az + Bv \tag{1.1}$$

Here y, z are the n-dimensional vectors of the phase coordinates of the controlled objects; u, v are the r-dimensional vectors of the controlling forces; A, B are constant matrices describing the system. All the vectors considered are treated as column vectors. An asterisk * represents a transposition. In agreement with [1], it is assumed that the control resources $u(t)$ and $v(t)$ which can be used for $t \geq \tau$ at each instant of time τ, are constrained by the condition

$$\rho_\tau [u(t)] \leqslant \mu(\tau), \qquad \rho_\tau [v(t)] \leqslant \nu(\tau) \tag{1.2}$$

We shall assume, that for any $\vartheta > \tau$ the quantity $\rho_\tau [w(t)]$, corresponding to functions $w(t)$ satisfying the condition $w(t) = 0$ when $t > \vartheta$, can be interpreted as the norm $\rho_{\tau,\vartheta} [w(t)]$ of the linear functional $\varphi_w [h(t)]$, generated by the function $w(t)$ on the adequate normed space $\{h(t)\}$ of the functions $h(t)$ $(\tau \leqslant t \leqslant \vartheta)$ (see, for instance, a similar case in [2], p . 6 and 7). Thus we shall limit ourselves to conditions (1.2) which eliminate discintinuities of $y(t)$ and $z(t)$.

The above conditions are satisfied for instance by constraints (1.2) of the form

$$\| u(t) \| \leqslant \mu, \qquad \| v(t) \| \leqslant \nu \qquad (\mu > \nu = \text{const}) \tag{1.3}$$

or the conditions

$$\left[\int_\tau^\infty \|u(t)\|^2 dt\right]^{1/2} \leqslant \mu(\tau), \qquad \left[\int_\tau^\infty \|v(t)\|^2 dt\right]^{1/2} \leqslant \nu(\tau) \tag{1.4}$$

but are not satisfied by constraints of impulsive type

$$\int_\tau^\infty \|u(t)\| dt \leqslant \mu(\tau), \qquad \int_\tau^\infty \|v(t)\| dt \leqslant \nu(\tau) \tag{1.5}$$

since the constraints (1.5) allow controls $u(t)$ and $v(t)$ which can include instantaneous δ-functions : $\delta(t - t_*)$, which would lead to discontinuities of the phase vectors $y(t)$ and $z(t)$. (In (1.3) to (1.5) and further on, the symbol $\|w\|$ represents the Euclidian norm of the vector w).

It is assumed, that a variation in the quantities $\mu(\tau)$ and $\nu(\tau)$ with variations of the time τ is obtained by a consumption of resources. For instance, if for $\tau \leq t < \vartheta$ the controls $u_*(t)$ and $v_*(t)$ were obtained, then in (1.4)

$$\Delta\mu^2 = \mu^2(\vartheta) - \mu^2(\tau) = -\int_\tau^\vartheta \|u_*(t)\|^2 dt, \qquad \Delta\nu^2 = \nu^2(\vartheta) - \nu^2(\tau) = -\int_\tau^\vartheta \|v_*(t)\|^2 dt$$

We shall consider the problem [1] of the minimax of time T elapsing before the encounter of the motions $y(t)$, $z(t)$. It is assumed that the goal of the pursuit is reached at the instant of encounter $t = \tau + T$ for which all the coordinates $y_i(t)$ and $z_i(t)$ $(i = 1, \ldots, n)$ coincide. Thus, we have a problem of $\min_u \max_v T = \max_v \min_u T$ with the condition $y(\tau + T) = z(\tau + T)$ and the assumption that the controls u and v at any instant $t = \tau$ are obtained according to the feedback principle for realizable values of $y(\tau), z(\tau), \mu(\tau), \nu(\tau)$, i.e. in the form of functions $u[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$ and $v[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$.

We shall assume that the systems (1.1) are fully controllable [3].

The assumption of full controllability does not limit the generality. In fact, if the systems are not fully controllable, the question of the encounter of the motions (1.1) has a meaning only when the difference $x(\tau) = y(\tau) - z(\tau)$ of the vectors $y(\tau)$ and $z(\tau)$ belongs to the subspace X , generated by the column vectors of the matrix $\{B, AB, \ldots, A^{n-1}B\}$.

Otherwise, it would not be possible to find controls $u(t)$ and $v(t)$ $(t \geq \tau)$ which would yield an encounter of the motions $y(t)$ and $z(t)$ at a finite value of time $\vartheta > \tau$. The validity of this statement results from the general theory of control by linear objects (see, for instance [2 and 3]). In the space X , the systems (1.1) are fully controllable. There follows, that when the systems (1.1) are not fully controllable in the original n-dimensional phase space of the vectors y and z , the problem of the encounter of the motions $y(t)$ and $z(t)$ can be reduced to the problem of the encounter of those motions in a phase space X of lower order, where those systems are fully controllable.

2. In the general case the solution of the conflicting problem of the minimax time occurring before the encounter of the two motions (and even the statement of the problem) meets some serious difficulties (see [1 and 4]). Thus, for a particular range of problems in the particular case described in Section 1, it is possible to formulate a simple rule determining a rational method for choosing the controls u and v . This rule has a simple intuitive meaning formulated as follows.

We shall compare the problem of pursuit (1.1) with the following problem of optimum quick response [5 and 6]. Find the control $w(t)$ constrained by the condition

$$\rho_\tau [w(t)] \leqslant \zeta(\tau) \tag{2.1}$$

and transferring the system

$$dx / dt = Ax \quad Bw \tag{2.2}$$

from the position $x = x(\tau)$ to the position $x(\tau + T) = 0$ in the smallest possible time $T = T^0$, i.e. it is necessary to determine

$$T^\circ = \min \quad \text{for} \quad x(\tau + T^\circ) = 0, \qquad \rho_{\tau,\, \tau+T^\circ} [w] \leqslant \zeta(\tau) \tag{2.3}$$

If $T^\circ [x(\tau), \zeta(\tau)]$ and $w_\tau^\circ(t)$ represent the solution of this problem, then for the original pursuit problem we take

$$T = T^\circ [y(\tau) - z(\tau), \ \ \mu(\tau) - \nu(\tau)] = \min\nolimits_u \max\nolimits_v T,$$

and we determine the optimum control with the equations

$$u^\circ [y(\tau), z(\tau), \mu(\tau), \nu(\tau)] = \frac{w_\tau^\circ(\tau)\, \mu(\tau)}{\mu(\tau) - \nu(\tau)}$$
$$v^\circ [y(\tau), z(\tau), \mu(\tau), \nu(\tau)] = \frac{w_\tau^\circ(\tau)\, \nu(\tau)}{\mu(\tau) - \nu(\tau)} \tag{2.4}$$

Here it is assumed that $\mu(\tau) > \nu(\tau)$, for the realizable values of $x(\tau) = y(\tau) - z(\tau)$, $\zeta(\tau) = \mu(\tau) - \nu(\tau)$ there exists a finite solution $T^\circ [x(\tau), \zeta(\tau)]$ of the problem of quick response (2.2),(2.3). Furthermore, we shall denote by the symbol G the domain

$$\zeta > 0, \qquad T^\circ [x, \zeta] < \infty \tag{2.5}$$

of the space $\{x, \zeta\}$ in which the problem (2.2),(2.3) has a solution for $\zeta(\tau) = \zeta$, $x(\tau) = x$

We shall illustrate the meaning of the formulated rule with the example of pursuit problem for which conditions (1.2) of the type (1.3) are present. Furthermore, we shall limit ourselves to the simplest case for which $r = n$ and the matrix D is nonsingular, i.e. the case in which the dimensions of the vectors y, z and u, v coincide. The difference $x(t) = y(t) - z(t)$ satisfies (2.2) where $w(t) = u(t) - v(t)$. Let us assume that at some instant $t = \tau$ the quantity $x(\tau) = y(\tau) - z(\tau)$ has been realized, and that the values $\zeta = \mu - \nu$, $x = x(\tau)$ are in the domain G (2.5). Here ζ is a constant. If for $t \geq \tau$, both controls u and v are in agreement with the equality (2.4) at every instant before the encounter, i.e. if

$$u(t) = u^\circ(t) = \frac{w_t^\circ(t)\, \mu}{\mu - \nu}, \qquad v(t) = v^\circ(t) = \frac{w_t^\circ(t)\, \nu}{\mu - \nu} \tag{2.6}$$

then the conditions (1.3) will be fulfilled, and in Equation (2.2) for $t \geq \tau$ we have $w(t) = w_\tau^\circ(t)$. Then there follows from the meaning of $w_\tau^\circ(t)$ the equality $x(t) = 0$, i.e. the encounter of the motions $y(t)$ and $z(t)$ will occur for the first time for $t = \tau + T^\circ [x(\tau), \zeta]$.

We shall assume now, that the control $v(t)$ for $t \geq \tau$ is modified continuously until the encounter, according to Equation (2.6), and that the control $u(t)$ is chosen arbitrarily, keeping in mind the constraint (1.3). Let us examine the function $V[x] = T^\circ[x, \zeta]$. This function is definite for all x and ζ of the domain G, positive definite in G, differentiable for $x \neq 0$, and for the problem (2.2),(2.3) represents an optimum Liapunov function [6]. This last condition mean that the derivative $(dV/dt)_w$ of the function $V[x(t)]$ satisfies Bellman's equation [6 and 7] along the motion $x(t)$ of the system (2.2) when controlled by $w(t)$

$$\min\nolimits_w \left(\frac{dV}{dt}\right)_w = \left(\frac{dV}{dt}\right)_{w^*} = -1 \tag{2.7}$$

Since

$$\left(\frac{dV}{dt}\right)_w = [\text{grad } V]^* [Ax + Bw] \tag{2.8}$$

then from (1.3), (2.1) and (2.7) there follows [6]

$$w_t^\circ(t) = -\frac{B^* [\text{grad } V] \zeta}{\| B^* [\text{grad } V] \|} \tag{2.9}$$

Computing the derivative $(dV/dt)_w$ for $w = u - v^\circ(t)$, then one can check the validity of the inequality

$$\left(\frac{dV}{dt}\right)_{u-v^\circ} \geqslant -1 \tag{2.10}$$

Integrating the inequality with respect to time for $t \geq \tau$ we get the inequality

$$V[x(t)] \geqslant V[x(\tau)] - (t-\tau) = T^\circ [x(\tau), \zeta] - (t-\tau) \tag{2.11}$$

the right-hand side of which is positive for $t < \tau + T^\circ$. But this means that for the choice $v = v^\circ [x(t), \zeta]$ according to (2.6), the encounter of the motions $y(t)$ and $z(t)$, i.e. the equality $x(t) = 0$, cannot occur for $t < \tau + T^\circ$, because the equality $V[x] = 0$ must be satisfied for $x = 0$ (It has been implicitly assumed here that during the motion, the point $x(t) = y(t) - z(t)$ does not leave the domain G).

If the point $x(t)$ leaves the domain G for some intervals of time, then the rule (2.4) cannot be used during those intervals. However, by choosing $v(t)$ arbitrarily $(\|v(t)\| \leq \nu)$, in those intervals of time, we can find that in such a case the encounter of $y(t)$ and $z(t)$ does not happen for $t < \tau + T^\circ [x(\tau), \zeta]$, if only the control of the motion $z(t)$ follows the rule (2.4) in the domain G.

On the contrary, if for $t \geq \tau$ it is assumed that $u = u^\circ(t)$ (2.6), we get the inequality

$$\left(\frac{dV}{dt}\right)_{u^\circ-v} \leqslant -1 \tag{2.12}$$

from which there follows that the point $x(t) = y(t) - z(t)$ does not leave the domain G before the encounter.

The integration of (2.12) yields the inequality

$$V[x(t)] \leqslant V[x(\tau)] - (t-\tau) = T^\circ [x(\tau), \zeta] - (t-\tau) \tag{2.13}$$

From the inequality (2.13), there follows that for $u = u^\circ(t)$ (2.6) the encounter of $y(t)$ and $z(t)$ occurs not later than for $t = \tau + T^\circ$, since for $x \neq 0$ we have $V[x] > 0$.

Thus, the control (2.4) in the considered case, actually guarantees the minimax of the time of the encounter. Here, that minimax coincides with the maximum and the game [8] corresponding to the problem of pursuit has a saddle point $T^\circ = T_{u^\circ, v^\circ}$. Consequently, the use of the general rule formulated above, turns out to be justified in the present case.

3. In the general case, a rigorous basis for the rule given in Section 2, meets with serious difficulties. Furthermore, it is possible to have situations for which either the rule turns out to be wrong, or cannot be utilized because for the realizable values of $\zeta(\tau) = \mu(\tau) - \nu(\tau)$ and $x(\tau) = y(\tau) - z(\tau)$, the problem (2.2),(2.3) does not have a finite solution $T^\circ[x(\tau), \zeta(\tau)]$. However this does not affect the validity of the formulated rule, since it can serve as an indicator for the choice of an optimum control u_0 and v_0 for a rather large class of cases.

We shall point out two difficulties which are met when substantiating the formulated rule. For that purpose, we shall discuss, for instance, the pursuit problem in the case of multidimensional objects $y(t)$ and $z(t)$ controlled by scalar controls u and v which are restricted by the constraints (1.3). It seems to be more difficult to make the proof following the scheme described in Section 2 than in the case $r = n$ considered in Section 2, since the function $V[x] = T^\circ[x, \mu - \nu]$ is not smooth any more. Because of the nonsmoothness of the function $V[x]$, the derivation and use of relations of the type (2.10) to (2.13) requires an additional analysis (see the analogous case in the investigation of the problem of optimum control in [9]). Another serious case which complicates the investigation is the question of the class of the controls $u(t)$ and $v(t)$ which can be realized in the system (1.1) when one of the partners follows the rule (2.4) and the other diverges from it. (If both partners follow the rules (2.4), the class of functions $u^{\circ}(t)$ and $v^{\circ}(t)$ (2.6) is determined by the class of elements $w(t)$ in the functional space $\{w(t)\}$ which has the norm $\rho_{\tau,\, \tau+T^{\circ}}(w(t))$ (see above page 263).

In the example, considered in [1] (page 12), it is shown that in the pursuit problem which follows the rule $u = u^{\circ}(t)$ in agreement with (2.6), in the case of the constraint (1.3) for $r < n$ (there $n = 2$, $r = 1$) slipping conditions can appear (there for $v(t) \equiv 0 \neq v^{\circ}(t)$). Consequently, in such cases, the statement of the pursuit problem must allow, in agreement with the feedback principle, realizations of the controlling signals $u(t)$ and $v(t)$ of a more general nature than the class of elements $w(t)$ of the space $\{w(t)\}$ which has the norm $\rho_{\tau}[w]$.

The conditions pointed out, as well as some other facts which are considered below in Sections 6 and 8 justify the expediency of the investigation of the general rule formulated in Section 2 for different concrete classes of the constraints (1.2).

The purpose of the present paper is to investigate the rule (2.4) in the case of constraints (1.2) of the form (1.4). This investigation constitutes the topic of Sections 5 and 9.

4. Let us consider the pursuit problem formulated in Section 1. We shall assume that the control resources are constrained by the conditions (1.4).

This means that, beginning from any instant of time $t = \tau$, only the controls $u(t)$ and $v(t)$ limited by the constraints (1.4) can be realized in the systems (1.1); whereupon if the controls $u_*(t)$ and $v_*(t)$ were realized for $\tau \leqslant t \leqslant \vartheta$, then

$$\mu^2(\vartheta) = \mu^2(\tau) - \int_{\tau}^{\vartheta} \|u_*(t)\|^2\, dt, \qquad \nu^2(\vartheta) = \nu^2(\tau) - \int_{\tau}^{\vartheta} \|v_*(t)\|^2\, dt \tag{4.1}$$

There follows that if at some instant t, the functions $u(t)$ and $v(t)$ are continuous, then

$$\frac{d\mu}{dt} = -\frac{\|u(t)\|^2}{2\mu}, \qquad \frac{d\nu}{dt} = -\frac{\|v(t)\|^2}{2\nu} \tag{4.2}$$

Let us specify the statement of the problem from the point of view of the class of permissible realizations of $u(t)$ and $v(t)$. We shall say that the control

$$u = u\,[y(t),\, z(t),\, \mu(t),\, \nu(t)] \tag{4.3}$$

is permissible if, for any arbitrary function $v(t)$, satisfying (4.1), Equation (4.3) determines a continuous realization of $u(t)$ satisfying (4.1), whereupon the realizations of $y(t)$, $z(t)$ and $\mu(t)$, $\nu(t)$ are solutions of the differential equations (1.1) and (4.2) (at least until the quantities $y(t)$, $z(t)$, $\mu(t)$, $\nu(t)$ remain in the domain in which the function (4.3) is definite). In an analogous manner the permissible control

$$v = v\,[y(\tau),\, z(\tau),\, \mu(\tau),\, \nu(\tau)] \tag{4.4}$$

can be determined.

Let us mention finally that the permissible controls (4.3) and (4.4) are mutually permissible, if they generate continous realizations of the controls $u(t)$ and $v(t)$, whereupon the realizations of $y(t)$, $z(t)$, $\mu(t)$ and $\nu(t)$ satisfy the differential equations (1.1) and (4.2) (in the domain in which the functions (4.3) and (4.4) are definite). We shall say that the mutually permissible controls (4.3) and (4.4) close the system (1.1) with a differential feedback loop. Later on, we shall investigate the original pursuit problem (1.1),(1.4) for controls (4.3) and (4.4) which close the system (1.1) with a differential feedback loop.

5. We shall show in this section, that in the case of a constraint (1.4) the rule (2.4) determines the permissible controls u (4.3) and v (4.4) which close the system (1.1) by a differential feedback loop. The following statement is valid.

L e m m a 5.1 . In the neighborhood of each point $x = x(\tau)$ and $\zeta = \zeta(\tau) > 0$, in which the problem (2.2),(2.3) has a finite solution $T^0[x,\zeta]$, the quantities $T^0[x,\zeta]$ and $w_T{}^0(t)$ are continuous in x and ζ .

P r o o f . In [6], it is shown, that the quantity $T^0[x,\zeta]$ is determined from Equation

$$\gamma(T) = \min_l \left[\int_0^T \| B^* F^{-1}(t)^* \, l \|^2 \, dt \right] = \frac{1}{\zeta^2} \qquad \text{for } x^* l = -1 \tag{5.1}$$

The optimum control $w_T{}^0(t)$ is determined by Equation

$$w_T{}^0(t) = B^* F^{-1}(t)^* l^0 \tag{5.2}$$

in which l^0 is a vector proportional to the solution of the problem (5.1). The vector l^0 is determined from Equation

$$\left[\int_0^T F^{-1}(t) BB^* F^{-1}(t)^* \, dt \right] l^0 = -x \tag{5.3}$$

the determinant of which

$$\Delta \left[\int_0^T F^{-1}(t) BB^* F^{-1}(t)^* \, dt \right]$$

is different from zero for all $T > 0$ when the conditions of full controllability are met [3 and 6].

Here $F^{-1}(t)$ is the inverse of the fundamental matrix $F(t)$ of the system $dx/dt = Ax$. The function $\gamma(T)$ in the left-hand side of (5.1) increases strictly with T , since the integrand of (5.1) can become zero when $l \neq 0$ only in separate points t . Similarly, the quantity γ is continuous in x . Thus, on the basis of a theorem on implicit functions, we conclude that Equation (5.1) determines a function $T^0[x,\zeta]$ continuous in x and ζ . This proves the lemma.

From Lemma 5.1 there follows that the functions u^0 and v^0 determined by Equations (2.4) are continuous in $y(\tau)$, $z(\tau)$, $\mu(\tau)$ and $\nu(\tau)$ of the open domain G (2.5) of the space $x = y - z$, $\zeta = \mu - \nu$. Consequently, substituting the quantities

$$u = u^0 [y(t), z(t), \mu(t), \nu(t)] = \frac{w_t{}^0(t)\,\mu(t)}{\mu(t) - \nu(t)} \tag{5.4}$$

$$v = v^0 [y(t), z(t), \mu(t), \nu(t)] = \frac{w_t{}^0(t)\,\nu(t)}{\mu(t) - \nu(t)} \tag{5.5}$$

in Equations (1.1), one obtains a complete set of Equations (1.1),(4.2), the right-hand sides of which are continuous in the domain G . Therefore, the system (1.1),(4.2),(5.4),(5.5) has in the domain G the continuous solution

$y(t)$, $z(t)$, $\mu(t)$, $\nu(t)$ which extends to the boundaries of the domain. An analogous conclusion is valid also in those cases in which only one of the controls u or v is determined by Equations (5.4) or (5.5), and the other control is chosen in the form of an explicit continuous function of time. But that means that the following statement is valid.

T h e o r e m 5.1 . The permissible controls u° (5.4) and v° (5.5) in the domain G (2.5) close the system (1.1),(4.2) by a differential feedback loop.

6. Let us now consider the problem of the optimum of the controls u° (5.4) and v° (5.5) in the sense of the original problem (1.1),(1.4) of the minimax time elapsing before the encounter.

First of all, it can be shown, as in Section 2, that when both partners follow the rules (5.4) and (5.5) for $t \geq \tau$, the encounter of the motions $y(t)$ and $z(t)$ occurs at the instant $t = \tau + T^\circ [y(\tau) - z(\tau),\ \mu(\tau) - \nu(\tau)]$, in other words the time T elapsing before the encounter is equal to $T^\circ[x(\tau),\ \zeta(\tau)]$ for any initial position $x(\tau) = y(\tau) - z(\tau)$ and $\zeta(\tau) = \mu(\tau) - \nu(\tau) > 0$, for which the time optimum control problem (2.2),(2.3) has the solution $T^\circ < \infty$.

However, that case presents little interest, since the most interesting cases in the pursuit problems are those for which one of the partners does not follow the standard behavior.

Let us assume now that the control u is always chosen in the form of the function (5.4), i.e. in agreement with the rule (2.4) and the control v is realized in the form of some continuous function $v(t)$ which satisfies the conditions (1.4) and (4.1). We shall assume that the process is considered from the instant $t = \tau$, and that the values of $y(\tau)$, $z(\tau)$, $\mu(\tau)$, $\nu(\tau)$ at that instant are such, that the point $x = x(\tau) = y(\tau) - z(\tau)$, $\zeta = \zeta(\tau) = \mu(\tau) - \nu(\tau)$ is in the domain G. It can be shown, that in that case the encounter of the motions $y(t)$ and $z(t)$ must occur not later than at the instant $t = \tau + T^\circ [x(\tau),\ \zeta(\tau)]$.

Let us consider the variations of $V(t) = T^\circ [x(t), \zeta(t)]$ as a function of time. As pointed out earlier in Section 5, the function $T^\circ[x, \zeta]$ is continuous in the domain G, and, consequently the function $V(t) = T^\circ [x(t), \zeta(t)]$ varies continuously with time along the continuous motions of the system (1.1),(4.2). Let us assume that at some instant $t > \tau$, for which the motions (1.1),(4.2) have not yet left the domain G, there are such values $x = x(t) = y(t) - z(t)$, $\zeta = \zeta(t) = \mu(t) - \nu(t)$, for which the problem

$$\gamma(T) = \min_l \left[\int_0^T \| B^* F^{-1}(\vartheta)^* l \|^2 \, d\vartheta\right] = \frac{1}{\zeta^2(t)} \quad \text{for } x^*(t)\, l = -1 \tag{6.1}$$

has a solution $l = l_0(t)$, $T = T^\circ(t)$, satisfying the condition

$$\| B^* F^{-1}(T)^* l_0(t) \| > 0 \tag{6.2}$$

Then, in that point x, ζ, T, the function $\gamma(T)$ has a positive derivative $\partial\gamma/\partial T$, and from the theorem on implicit functions we deduce, that the function $T^\circ[x, \zeta]$ is differentiable in the neighborhood of the point $x = x(t)$, $\zeta = \zeta(t)$. Therefore, we can calculate the derivative $(dV(t)/dt)_{u^\circ, v}$

at the point $x(t)$, $\zeta(t)$ on the basis of Equations (1.1) and (4.2) for $u = u^\circ$ (5.4), $v = v(t)$. We get

$$\left(\frac{dV(t)}{dt}\right)_{u^\circ - v} = \sum_{i=1}^{n} \frac{\partial T^\circ}{\partial x_i}\left[\sum_{j=1}^{n} a_{ij}x_j + \sum_{k=1}^{r} b_{ik}(u_k^\circ - v_k)\right] + \frac{\partial T^\circ}{\partial \zeta}\left[-\frac{\|u^\circ\|^2}{2\mu} + \frac{\|v\|^2}{2\nu}\right] \quad (6.3)$$

where a_{ij} and b_{ik} are elements of the matrices A and B. Simultaneously the function $W(t) = T^\circ[x(t), \zeta(t)]$, calculated on the motions $x(t)$, $\zeta(t)$ of the system (2.2), for $w = w(t)$, has at the same point $x = x(t)$, $\zeta = \zeta(t)$, a derivative $(dW/dt)_w$ satisfying Bellman's equation

$$\min_u \left(\frac{dW}{dt}\right)_w = \left(\frac{dW}{dt}\right)_{w^\circ} = \sum_{i=1}^{n} \frac{\partial T^\circ}{\partial x_i}\left[\sum_{j=1}^{n} a_{ij}x_j + \sum_{k=1}^{r} b_{ik}\overset{\circ}{w}_{t,k}(t)\right] + \frac{\partial T^\circ}{\partial \zeta}\left(-\frac{\|w_t^\circ(t)\|^2}{2\zeta}\right) = -1 \quad (6.4)$$

since the function $T^\circ[x, \zeta]$ is an optimum Liapunov function for (2.2), and the function $w_t^\circ(t)$ is an optimum control. The quantity

$$\left(\frac{dW}{dt}\right)_w = \sum_{i=1}^{n} \frac{\partial T^\circ}{\partial x_i}\left[\sum_{i=1}^{n} a_{ij}x_j + \sum_{k=1}^{r} b_{ik}w_k\right] + \frac{\partial T^\circ}{\partial \zeta}\left(-\frac{\|w\|^2}{2\zeta}\right)$$

has a minimum for $w = w_t^\circ(t)$. Therefore Equations

$$\sum_{i=1}^{n} \frac{\partial T^\circ}{\partial x_i} b_{ik} - \frac{\partial T^\circ}{\partial \zeta}\frac{\overset{\circ}{w}_{t,k}(t)}{\zeta} = 0 \quad (6.5)$$

are satisfied.

Substituting the value u° (5.4) in (6.3) and assuming $v = v^\circ + \partial v$, where v° is determined by the equality (5.5) we get the following equation by taking (6.4) and (6.5) into consideration:

$$\left(\frac{dV(t)}{dt}\right)_{u^\circ - v} = -1 + \frac{\partial T^\circ}{\partial \zeta}\frac{\|\delta v\|^2}{2\nu} \leqslant -1 \quad (6.6)$$

since there follows from (5.1) that $\partial T^\circ/\partial\zeta < 0$ when $\gamma(T)$ increases. Thereby if $\delta v \not\equiv 0$, the strict inequality is satisfied in (6.6).

Thus, the inequality (6.6) is satisfied at the point $x(t) = y(t) - z(t)$, $\zeta(t) = \mu(t) - \nu(t)$

If the values of $x(t)$ and $\zeta(t)$ can be found such that the solution $l = l_0(t)$ and $T = T^\circ(t)$ of the prblemm (6.1) does not verify the condition (6.2), that is for which the equality

$$\|B^* F^{-1}(T)^* l_0(t)\| = 0$$

is satisfied, then the calculation of the derivative $(dV/dt)_{u-v}$ becomes more difficult since the theorem on the differentiability of the implicit function $T^\circ[x, \zeta]$ cannot be used any more. To study the behavior of the function $V(t)$ in the neighborhood of such points $x = x(t)$ and $\zeta = \zeta(t)$ we shall associate with the system (2.2) the auxiliary system

$$dx / dt = Ax + Bw + \varepsilon Es \quad (6.7)$$

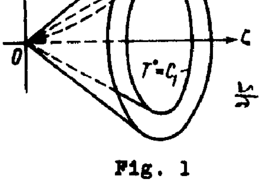

Fig. 1

for which $\varepsilon > 0$ is a small parameter. Here E is the unit matrix and s is the n-dimensional vector of the complementary control. The problem of the limit time optimum control of the system (6.7)

$$(x(\tau) \to x(\tau + T_\varepsilon^\circ) = 0, \ T_\varepsilon^\circ = \min)$$

with the constraints

$$\int_\tau^\infty \|\{w(t),\ s(t)\}\|^2 dt \leqslant \zeta^2(\tau) \quad (6.8)$$

has the solution $T_\varepsilon^\circ[x(\tau), \zeta(\tau)], \{w_\tau^{(\varepsilon)}(t), s_\tau^{(\varepsilon)}(t)\}$ for all $x = x(\tau)$ and $\zeta = \zeta(\tau)$ of the domain G (2.5), in which the problem (2.2),(2.3) has a solution. The function $T_\varepsilon^\circ[x, \zeta]$ is differentiable in the neighborhood of any point $x = x(\tau)$, $\zeta = \zeta(\tau)$ where the problem (6.7),(6.8) has a finite solution $T_\varepsilon^\circ[x, \zeta] > 0$, i.e. in any case, everywhere in the domain G.

Furthermore, in the closed neighborhood of each point x, ζ of G the following limit relations

$$\lim T_\varepsilon^\circ[x, \zeta] = T^\circ[x, \zeta], \quad \lim w_\tau^{(\varepsilon)}(t) = w_\tau^\circ(t), \quad \lim s_\tau^{(\varepsilon)}(t) = 0 \quad \text{при } \varepsilon \to 0 \tag{6.9}$$

are satisfied uniformly.

Let us compare the variation of the functions $V(t) = T^\circ[x(t), \zeta(t)]$ and $V_\varepsilon(t) = T_\varepsilon^\circ[x(t), \zeta(t)]$ along the motion of the system (1.1), (1.4) for small time intervals $\Delta t > 0$, when $w = u^\circ - [v^\circ + \delta v]$, $s = 0$. The function $T_\varepsilon^\circ = V_\varepsilon$ satisfies Bellman's equation

$$\min_{\{w,s\}} (dV_\varepsilon / dt) = (dV_\varepsilon / dt)_{\{w^\varepsilon, s^\varepsilon\}} = -1$$

on the basis of the system (6.7). Using this equation, the limit conditions, (6.9), and estimating the derivative $(dV_\varepsilon / d\vartheta)_{u^\circ - v, s=0}$ for $t \leqslant \vartheta \leqslant t + \Delta t$ in a manner similar to that used above for the function V (see (6.3) to (6.6)), we get the inequality

$$V(t + \Delta t) - V(t) \leqslant -\Delta t\,(1 + \varkappa \|\delta v\|^2 - O(\varepsilon)) \tag{6.10}$$

where $\varkappa > 0$ is a constant and the symbol $O(\varepsilon)$ denotes an infinitesimally small value for $\varepsilon \to 0$.

From (6.10) there follows that for $u = u^\circ$, the inequality

$$\sup \left(\frac{dV}{dt}\right)^+_{u^\circ - v} \leqslant -1 \tag{6.11}$$

is satisfied for any control $v(t)$. Here the symbol $\sup (dV/dt)^+_{u^\circ - v}$ denotes the highest right-hand side value of a derivative of the function $V(t)$ at the point $x(t)$, $\zeta(t)$.

Thus, we come to the conclusion, that for $u = u^\circ$ and any choice of the control $v(t)$ (1.4), the inequality (6.11) is satisfied for all the values $t \geq \tau$, for which the motion $x(t) = y(t) - z(t)$, $\zeta(t) = \mu(t) - \nu(t)$ of the system (1.1),(4.2) still remains in the domain G, determined by the inequalities

$$\mu(t) - \nu(t) > 0, \quad \mu(t) > 0, \quad \nu(t) > 0, \quad T^\circ[x, \zeta] < \infty \tag{6.12}$$

where the function $V(t) = T^\circ[x(t), \zeta(t)]$ changes continuously with the variations of t.

In the domain G (2.5) the function $T[x,\zeta]$ is positive definite everywhere, except on the axis $x = 0$. The level surfaces $T^\circ[x,\zeta] = \text{const} > 0$ in the space $\{x,\zeta\}$ are cones, the intersection of which by surfaces $\zeta = \text{const} > 0$ are ellipses (Fig.1).

But in such a case the inequalities (6.6) and (6.11) mean that the motions of the system (1.1),(4.2) for $u = u^\circ$ (5.4) and at any instant of time $t \geq \tau$, as long as such motions remain in the domain G, intersect the surfaces $T^\circ[x,\zeta] = \text{const} = \sigma$ in the direction of decreasing of σ, i.e. from the outside to the inside. It follows, that for $t \geq \tau$, the motion $x(t) = y(t) - z(t)$, $\zeta(t) = \mu(t) - \nu(t)$ remains in the domain G as long as $\|x\| > 0$. It follows that, by virtue of (6.6) and (6.11), the inequality

$$\tau + T^\circ[x(\tau), \zeta(\tau)] \geqslant t + T^\circ[x(t), \zeta(t)] \tag{6.13}$$

is satisfied for all $t \geq \tau$, as long as $\|x(t)\| > 0$.

Since for all $\|x\| > 0$, we have $T^\circ[x,\zeta] > 0$, it follows from (6.13) that for $u = u^\circ$ (5.4) the equality $x(t) = 0$, or in other words the encounter of the motions $y(t)$ and $z(t)$ occurs no later than the instant $t = \tau + T^\circ[x(\tau), \zeta(\tau)]$.

Thus the following statement is valid.

T h e o r e m 6.1. Let the point $x(\tau) = y(\tau) - z(\tau)$, $\zeta(\tau) = \mu(\tau) - \nu(\tau)$

be in the domain G (2.5), and let us assume that for $t \geq \tau$ the control $u = u(t)$ is always chosen in agreement with the equality (5.4). Then, for any continuous control $v = v(t)$, constrained by the relation (4.1), the encounter of the motions $y(t)$ and $z(t)$ will occur no later than at the instant

$$t = \tau + T^\circ[x(\tau), \zeta(\tau)]$$

We shall notice that if the control $v(t)$ differes from the optimum control $v^\circ(t)$ (5.5) for a set of values of t of the positive norm, on the interval $\tau \leqslant t \leqslant \tau + T^\circ[x(\tau), \zeta(\tau)]$ then for $u = u^\circ(t)$ (5.4) the encounter of the motions $y(t)$ and $z(t)$ occurs earlier than $t = \tau + T^\circ[x(\tau), \zeta(\tau)]$, since in such a case for values $t < \tau + T^\circ[x(\tau), \zeta(\tau)]$, (6.13) yields a strict inequality of the type

$$\tau + T^\circ[x(\tau), \zeta(\tau)] > t + T^\circ[x(t), \zeta(t)] + \varepsilon.$$

The theorem 6.1 shows that the control u°, determined by the equality (6.4) is optimum in the given sense for the tracking motion $y(t)$ (1.1). More precisely, from this theorem, and from the condition that for any control $u = u^*$, for a given choice of $v = u^*\nu/\mu$, the encounter of $y(t)$ and $z(t)$ does not occur before $t = \tau + T^\circ$. We note that $T^\circ = \min_u \max_v T = T_{u^\circ, v^\circ}$.

7. In this section we shall consider the situation which arises when the pursued motion $z(t)$ (1.1) is controlled by the condition (2.4).

Thus, we shall assume that at the moment $t = \tau$ we have the values $x(\tau) = y(\tau) - z(\tau)$, $\zeta(\tau) = \mu(\tau) - \nu(\tau)$, which lie in the domain G (2.5). Furthermore, for $t \geq \tau$ the control v is chosen equal to v° (5.5) at all instants at which the motion $x(t) = y(t) - z(t)$, $\zeta(t) = \mu(t) - \nu(t)$ remains in the domain G and until the encounter of the motions $y(t)$ and $z(t)$ occurs. In such a case, and for any given instant of time, the inequality

$$\inf \left(\frac{dT^\circ[x(t), \zeta(t)]}{dt}\right)^+_{u-v^\circ} \geqslant -1 \tag{7.1}$$

is satisfied. This inequality is derived in a manner similar to that used for deriving the inequalities (6.6) and (6.11). In the inequality (7.1) the symbol $\inf (dT^\circ / dt)^+_{u-v^\circ}$ represents the lowest right-hand side value of the derivative of the function $V(t) = T^\circ[x(t), \zeta(t)]$, calculated along the motions of the system (1.1),(4.2) for $v = v^\circ$ (5.5). Integrating the inequality (7.1) with respect to time for all the instants of time for which the motion $x(t)$, $\zeta(t)$ (1.1),(4.2) still remains in the domain G and $x(t) \neq 0$, we get the inequality

$$t - \tau \geqslant T^\circ[x(\tau), \zeta(\tau)] - T^\circ[x(t), \zeta(t)] \tag{7.2}$$

However, for further considerations in this section, unlike in Section 6, we must take into account the new circumstances, which complicate the solution of the pursuit problem. In the case $u = u^\circ$ (5.4), considered in Section 6, the inequalities (6.6),(6.11) guarantee the conservation of the motion $x(t) = y(t) - z(t)$, $\zeta(t) = \mu(t) - \nu(t)$ in the domain G (2.5) till the encounter of the motions $y(t)$ and $z(t)$. On the contrary here, the motion $x(t)$, $\zeta(t)$ for $v = v^\circ(t)$ and $u \neq u^\circ(t)$ can reach the boundaries of the domain G, before the encounter of $y(t)$ and $z(t)$

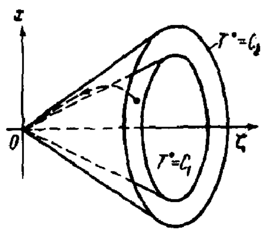

Fig. 2

When the motion $x(t)$, $\zeta(t)$ touches the boundary G at some instant of time $t = \vartheta$, one of the following relations

$$\lim [\mu(t) - \nu(t)] = 0 \qquad (7.3)$$

$$\lim T^\circ [x(t), \mu(t) - \nu(t)] = \infty \qquad (7.4)$$

$$\lim \mu(t) = 0 \qquad (7.5)$$

must be satisfied when $t \to \vartheta - 0$. If the boundary conditions (7.3) and (7.5) are satisfied, where at $t = \vartheta$, $x(\vartheta) \neq 0$, then for $t > \vartheta$ the two motions $y(t)$ and $z(t)$ will behave freely, independently from the action of the controls $u(t)$ and $v(t)$, the resources of which, according to (7.3) and (7.5) are exhausted at the instant $t = \vartheta$. But under such conditions the motions $y(t)$ and $z(t)$ do not encounter at all. If at the moment $t = \vartheta$ the boundary condition (7.5) is the only one satisfied, and $x(\vartheta) \neq 0$, then for $t > \vartheta$ it is always possible to consider the control $v(t)$ such that the encounter of $y(t)$ and $z(t)$ is unrealizable (for this, it is sufficient to assume $v(t) \equiv 0$ for $t > \vartheta$).

Thus, the control $u = u(t)$, for which (7.5) occurs at the instant $t = \vartheta$ but for which the encounter of the motions $y(t)$ and $z(t)$ does not occur, is not profitable for the pursuing object, and in the future we shall not consider such cases.

In the case in which the relations (7.3), or (7.4) are satisfied before the encounter of the motions $y(t)$ and $z(t)$ but the relation (7.5) is not satisfied, an even more confused situation occurs, where the rule (2.4) cannot be used in order to control by means of the motions (1.1), since then, the motion (1.1),(4.2) goes out of the domain G in which this rule is meaningful.

An analogous situation arises in those cases in which, right from the beginning of the process, quantities $x(\tau) = y(\tau) - z(\tau)$, $\zeta(\tau) = \mu(\tau) - \nu(\tau)$, which do not belong to the domain G are obtained for $t = \tau$. A comprehensive discussion of that case goes out of the frame of the present work.

We shall assume that the motion $x(t) = y(t) - z(t)$, $\zeta(t) = \mu(t) - \nu(t)$ does not leave the domain G for all $t \geq \tau$ before the encounter of the points $y(t)$ and $z(t)$. However, even in such a case, it is not possible to conclude from the inequalities (7.1) and (7.2) that the encounter of $y(t)$ and $z(t)$ will not occur earlier than at the instant $t = \tau + T^\circ[x(\tau), \zeta(\tau)]$. Consequently, although the functions $T^\circ[x,\zeta]$, $u^\circ[x,\zeta]$ and $v^\circ[x,\zeta]$ satisfy Bellman's equation

$$\min_u \max_v \left(\frac{dT^\circ}{dt}\right)_{u,\,v} = \max_v \min_u \left(\frac{dT^\circ}{dt}\right)_{u,\,v} = \left(\frac{dT^\circ}{dt}\right)_{u^\circ,\,v^\circ} = -1 \qquad (7.6)$$

in the domain G, even when the motions $x(t)$, $\zeta(t)$ do not leave that domain, the value T° and the pair of controls u°, v° do not have a corresponding meaning of $\max_v \min_u T$ and a saddle point $\{u^\circ, v^\circ\}$ for the game [8] which corresponds to the given pursuit problem. The conclusion is such that for $x(t) \to 0$ the quantity $T^\circ[x(t), \zeta(t)]$ can avoid going to zero, if $\zeta(t) \to 0$ (see Fig.2).

Such a situation occurs for instance in the following simple problem.

Example 7.1. We shall consider the systems (1.1) described by the first order equations

$$dy / dt = u, \qquad dz / dt = v \tag{7.7}$$

where y, z, u and v are scalars. In agreement with (4.2) the variations of the quantities

$$\mu^2(\tau) = \int_\tau^\infty u^2(t)\,dt, \qquad \nu^2(\tau) = \int_\tau^\infty v^2(t)\,dt$$

are described by Equations

$$\frac{d\mu}{dt} = -\frac{u^2}{2\mu}, \qquad \frac{d\nu}{dt} = -\frac{v^2}{2\nu} \tag{7.8}$$

In this case, the system (2.2) becomes

$$dx / dt = w \qquad (x = y - z, w = u - v) \tag{7.9}$$

Let $\tau < 0$. We shall try to choose the initial conditions $x(\tau) = y(\tau) - z(\tau)$ and $\zeta(\tau) = \mu(\tau) - \nu(\tau)$ and the control $u(t)$ $(\tau \le t < 0)$, such that for all time $\tau \le t < 0$, along the motion $x(t)$, (7.9) for which $u = u(t)$, $v = v(t)$ (5.5), the condition

$$\{x(t), \zeta(t)\} \in G, \qquad T^\circ[x(t), \zeta(t)] = 1 \tag{7.10}$$

is satisfied, and the encounter $y(t) = z(t)$ occurs for $t = 0$. In order to satisfy the conditions (7.10) it is sufficient, according to (5.1), that the condition

$$\zeta(t) = \mu(t) - \nu(t) = x(t) > 0 \tag{7.11}$$

be satisfied since in the given case we have $F(t) = 1$ in (5.1). Control $w_\tau^\circ(t)$, calculated according to (5.2) is

$$w_\tau^\circ(t) = -x(\tau) \tag{7.12}$$

and consequently, in agreement with (5.5), (7.11) and (7.12) we have the condition

$$v^\circ(t) = -\nu(t) \tag{7.13}$$

From (7.8) and (7.13), choosing $\nu(0) = 1$, we get

$$\nu(t) = e^{-1/2 t} \tag{7.14}$$

Then, from (7.9),(7.11),(7.13) and (7.14) it follows

$$\mu(t) = e^{-1/2 t} + x(t), \qquad u(t) = -e^{-1/2 t} + dx / dt \tag{7.15}$$

Now the function $x(t)$ is determined from the differential equation obtained by substituting (7.15) in the first equation (7.8); this equation has the form

$$dx/dt = -x(t) - \sqrt{x(t)e^{-1/2 t} + x^2(t)} \tag{7.16}$$

For the condition $x(0) = 0$, Equation (7.16) can be solved in the form of the series

$$x(t) = {}^1\!/_4 t^2 + \varphi(t), \qquad \varphi(t) = \alpha_3 t^3 + \alpha_4 t^4 + \ldots \tag{7.17}$$

whereupon we get for the function $\varphi(t)$, the differential equation

$$d\varphi / dt = f[t, \varphi]$$

which has a holomorphic right-hand side in the neighborhood of the point $t = 0$, $x = 0$. Thus, according to Cauchy's theorem [10] there follows the convergence of the series (7.17) for sufficiently small values of t. Furthermore, the function $x(t)$ (7.17) is positive for small values of t.

Thus, for sufficiently small values of $t(\tau \le t < 0)$ we design the control (7.15)

$$u(t) = -e^{-1/2 t} + {}^1\!/_2 t + d\varphi / dt$$

such, that although the control $v = v^\circ[x(t), \zeta(t)]$ is chosen for all $t \in [\tau, 0)$ always in agreement with (2.4), i.e. in the form (5.5), where $T^\circ[x(t), \zeta(t)] \equiv 1$ $(\tau \leqslant t < 0)$, i.e. $(dT^\circ / dt)_{u-v^\circ} \equiv 0$, yet the encounter of the motions $y(t)$ and $z(t)$ becomes real for $t = 0$. Consequently, in the given example, the time T elapsing before the encounter, for $v = v^\circ$, is smaller than $T^\circ[x(\tau), \zeta(\tau)]$.

This example proves the statement given above, that in general, the quantity $T^\circ[x(\tau), \zeta(\tau)]$ is not a minimax of the time elapsing before the encounter of $y(t)$ and $z(t)$, and that the pair of controls u° (5.4) and v° (5.5) does not represent a saddle point [9] of the corresponding game even if we limit ourselves to the controls u and v which do not move the motions (1.1),(4.2) out of the domain G (2.5).

More specifically, the example shows the possibility of such variations of the control u from u°, for which, although the second motion in (1.1) remains controlled according to (2.4) for all $t \geq \tau$, the encounter occurs earlier than at the instant $t = T^\circ[y(\tau) - z(\tau), \mu(\tau) - \nu(\tau)] + \tau$. However, it can occur inside the domain G only at the condition that $\zeta(t) = \mu(t) - \nu(t) \to 0$, but then $\lim T^\circ [x(t), \zeta(t)] \neq 0$.

The following statement is valid.

Theorem 7.1. Let us assume that at the instant $t = \tau$ we have the quantities $x(\tau) = y(\tau) - z(\tau)$, $\zeta(\tau) = \mu(\tau) - \nu(\tau)$, which are located in the domain G (2.5). If during the time t

$$\tau \leqslant t < \vartheta < \tau + T^\circ [x(\tau), \zeta(\tau)] \tag{7.18}$$

the motion $x(t) = y(t) - z(t)$, $\zeta(t) = \mu(t) - \nu(t)$ remains in the domain G, where the inequality $\zeta(t) > \varepsilon$ ($\varepsilon > 0$ is a constant), is satisfied, and the control $v = v(t)$ for $t \geq \tau$ is chosen in agreement with the equality (5.5), then the encounter of the motions $y(t)$ and $z(t)$ cannot occur at the instant $t = \vartheta$

Proof. The quantities $T^\circ [x(t), \zeta(t)]$, $x(t)$ and $\zeta(t)$ change continuously along the motion of the system (1.1),(4.2). Consequently, under the conditions of the theorem the inequality $\zeta(\vartheta) \geqslant \varepsilon > 0$ is satisfied at the instant $t = \vartheta$ If, in spite of the statement of the theorem, we take $x(\vartheta) = 0$, we must have

$$\lim T^\circ [x(t), \zeta(t)] = 0 \quad \text{for} \quad t \to \vartheta - 0 \tag{7.19}$$

since the quantity $T^\circ [x, \zeta(\vartheta)]$ is continuous and positive definite in x for $\zeta(\vartheta) \geqslant \varepsilon$ But the relations (7.18),(7.19) and (7.2) are contradictory; this contradiction proves the theorem.

Note. From the proof of Theorem 7.1. it can be seen that for $\vartheta < \tau + T^\circ [x(\tau), \zeta(\tau)]$ the condition $\lim T^\circ [x(t), \zeta(t)] = 0$ when $t \to \vartheta - 0$, cannot be satisfied if the point $x(t)$, $\zeta(t)$ does not leave the domain G, and the equality $v = v^\circ(t)$ is always satisfied.

8. The theorems 6.1 and 7.1 show that the choice of the control $v = v^\circ [x(t), \zeta(t)]$ (5.5) for the second motion (1.1) will be expedient in the domain G, at least until $\zeta(t) = \mu(t) - \nu(t) > \varepsilon$, where ε is some positive number arbitrarily chosen beforehand.

At the same time, the example 7.1 shows that for $\zeta(t) = \mu(t) - \nu(t) \to 0$, but $\lim T^\circ [x(t), \mu(t) - \nu(t)] \neq 0$ $(t \to \vartheta - 0)$, the motion $z(t)$, can be sometimes caught up by the motion $y(t)$, (for $t = \vartheta$) before the instant $t = \tau + T^\circ [x(\tau), \zeta(\tau)]$, if this motion $z(t)$ is also rigorously controlled according to (2.4) when $\zeta(t) < \varepsilon$. Thus, as far as the motion $z(t)$ is concerned when $\zeta(t) = \mu(t) - \nu(t) \to 0$ for $T^\circ [x(t), \zeta(t)] > \varepsilon > 0$ it would be worth while changing this motion to another control rule distinct from the rule (5.5).

The construction of such an optimum control method, which would be described by the function $v = v[y(t), z(t), \mu(t), \nu(t)]$ and which would take into account the indicated circumstances is a different complicated problem which is beyond the scope of this paper.

However, if one is to accept the possibility of additional information on the motion $y(t)$, then we can indicate a series of simple and rather expedient control methods by the motion $z(t)$. One such method follows.

We shall consider the problem of the pursuit of the second of the motions in (1.1) by the first, under conditions which facilitate the problem of the control by the pursued motion $z(t)$. We shall assume, as in the previous problem, that in the device generating the control u, the realized quantities $y(t)$, $z(t)$, $\mu(t)$, $\nu(t)$ can be taken into account at any instant t. However, we shall now assume, that in the device generating the control v, one can take into account, together with those values also the quantity $u(t-\eta)$, where $\eta > 0$ is a small constant quantity. In other words, we shall accept the possibility of choosing the control $v(t)$ in the form

$$v = v\,[y(t),\, z(t),\, \mu(t),\, \nu(t),\, u(t-\eta)]$$

Then we shall take for $u(t)$ and $v(t)$ some piece-wise continuous realizations. In such a case, in the situations pointed out in this section, we can assume

$$v(t) = u(t-\eta) \tag{8.1}$$

if the delay η is a sufficiently small quantity.

The control $v(t)$ (8.1) has the following property. If at the instant $t = \tau$ the quantities $x(\tau)$ and $\zeta(\tau)$ are obtained in the domain G, then, for a sufficiently small $\eta > 0$, the control $v(t)$ (8.1) guarantees the encounter of the motions $y(t)$ and $z(t)$ not earlier than at the instant $t > \tau + T^\circ[x(\tau), \zeta(\tau)] - \vartheta_*$, where ϑ_* is an arbitrarily small positive number chosen beforehand. If at the initial instant $t = \tau$ we have quantities $x(\tau)$ and $\zeta(\tau)$ outside the domain G (2.5), then for a sufficiently small $\eta > 0$, the control $v(t)$ (8.1) guarantees that the encounter of the motions $y(t)$ and $z(t)$ will not occur earlier than at $t > \tau + \theta_*$, where θ_* is an arbitrary large positive number chosen beforehand. We shall not carry out the investigation of those properties.

Taking Theorems 6.1 and 7.1 into account, and also the properties of the control $v(t)$ (8.1), we get the following rule for choosing the control v in an expedient manner. Let us choose some small numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. If at the instant t we have values $x(t) = y(t) - z(t)$ and $\zeta(t) = \mu(t) - \nu(t)$ satisfying the inequalities

$$\zeta(t) > \varepsilon_1, \qquad T^\circ[x(t), \zeta(t)] < \infty \tag{8.2}$$

then, the control v is chosen according to (5.5). If at the instant t one of the conditions (8.2) does not hold, where $T^\circ[x(t), \zeta(t)] > \varepsilon_2$, then the control v is chosen according to (8.1).

Then, in order to avoid the appearance of slipping modes in the case of frequent changes of the controls v (5.5) and (8.1) along the motion (1.1), (4.2), we can introduce a small hysteresis in the controls (5.5), (8.1). More precisely, it is possible to change from the control (5.5) to the control

(8.1) when the inequalities

$$\zeta(t) > \varepsilon_1^{(1)}, \qquad T^\circ[x(t), \zeta(t)] < \infty \tag{8.3}$$

do not hold any more, and to change from the control (8.1) to the control (5.5) when the inequalities

$$\zeta(t) \geqslant \varepsilon_1^{(2)}, \qquad T^\circ[x(t), \zeta(t)] \leqslant \theta \tag{8.4}$$

are satisfied, where $\varepsilon_1^{(2)} > \varepsilon_1^{(1)} > 0$ and θ is a sufficiently large number.

9. To conclude this paper we shall discuss the relation between the rule (2.4) used for choosing the controls $u = u^\circ$ (5.4) and $v = v^\circ$ (5.5) and the aiming rule formulated in [1] and based on guiding the motions $y(t)$ and $z(t)$ to the point $\xi^\circ(t)$, intersection of the boundaries (see Fig.3) of the domains of accessibility $H^{(1)}[y(t), \mu(t), t + T_0]$ and $H^{(2)}[z(t), \nu(t), t + T_0]$ of the processes $y(t)$ and $z(t)$ at the instant $t + T_0$ $[y(t), z(t), \mu(t), \nu(t)]$ at which the process $y(t)$ takes over the process $z(t)$ (see [1], pages 7 to 9).

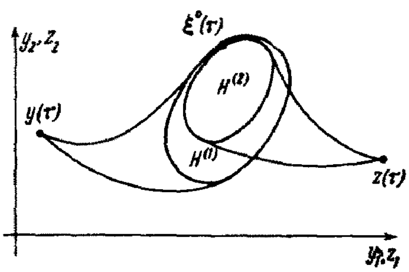

Fig. 3

For definiteness, we shall only consider here, as was done earlier in Sections 6 to 8, the case of boundaries (1.2) of the type (1.4). We shall show that the two rules coincide with one another.

First of all we shall notice that the equality

$$T^\circ[x(\tau), \zeta(\tau)] = T_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau)] \tag{9.1}$$

is valid, where both quantities have a meaning for the same values of

$$x(\tau) = y(\tau) - z(\tau), \quad \zeta(\tau) = \mu(t) - \nu(\tau)$$

In fact, let us assume that for some $x(\tau), \zeta(\tau)$ the quantity $T^\circ[x(\tau), \zeta(\tau)]$ has a meaning. This means that there exists a control $w_\tau^\circ(t)$, constrained by the condition

$$\rho_{\tau, \tau+T^\circ}[w_\tau^\circ(t)] \leqslant \zeta(\tau) \tag{9.2}$$

and which brings the system into the state $x(\tau + T^\circ) = 0$. But in such a case, for any given control $v(t)$, constrained by the condition

$$\rho_{\tau, \tau+T^\circ}[v(t)] = \left[\int_\tau^{\tau+T^\circ} \|v(t)\|^2\, dt\right]^{1/2} \leqslant \nu(\tau) \tag{9.3}$$

the control

$$u(t) = v(t) + w_\tau^\circ(t) \tag{9.4}$$

satisfies the condition

$$\rho_{\tau, \tau+T^\circ}[u(t)] = \left[\int_\tau^{\tau+T^\circ} \|u(t)\|^2\, dt\right]^{1/2} \leqslant \nu(\tau) + \zeta(\tau) = \mu(\tau) \tag{9.5}$$

and guarantees the encounter of the motions $y(t)$ and $z(t)$ at the instant $t = \tau + T^\circ$. And this means that the domain of accessibility $H^{(2)}[z(\tau), \nu(\tau), \tau + T^\circ]$ of the motion $z(t)$ is in the domain of accessibility $H^{(1)}[y(\tau), \mu(\tau), \tau + T^\circ]$ of the motion $y(t)$. Consequently, for those values of $x(\tau)$ and $\zeta(\tau)$ the quantity $T_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$ has a meaning, and the inequality

$$T_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau)] \leqslant T^\circ[x(\tau), \zeta(\tau)] \tag{9.6}$$

is satisfied.

Now let us assume the opposite, that is, for realized values of $y(\tau)$,

$x(\tau)$, $\mu(\tau)$ and $\nu(\tau)$, the quantity T_0 has a meaning. We shall consider the problem of the control by the system (2.2) from the point $x = x(\tau)$ to the point $x(\tau + T_0) = 0$ with the smallest possible norm

$$\rho_{\tau,\tau+T_0}[w(t)] = \left[\int_{\tau}^{\tau+T_0} \|w(t)\|^2 dt\right]^{1/2} = \zeta^\circ = \min \tag{9.7}$$

Under conditions of full controllability of the system (2.2) this problem has a solution. If in (9.7) we have

$$\zeta^\circ \leqslant \mu(\tau) - \nu(\tau) \tag{9.8}$$

it means that $T^0 \leq T_0$ and as a consequence of the inequality (9.6) the equivalence of the quantities T_0 and T^0 will be proved.

Let us assume now that in (9-7) we had

$$\zeta^\circ > \mu(\tau) - \nu(\tau) \tag{9.9}$$

Let $w_\tau(t)^\circ$ be an optimum control solving the problem of the control $x = x(\tau)$, $x(\tau + T_0) = 0$ for the system (2.2) with the condition (9.7). Then

$$x(\tau + T_0) = F(T_0)x(\tau) + \int_{\tau}^{\tau+T_0} F(\tau + T^* - t) Bw_\tau(t)^\circ dt = 0 \tag{9.10}$$

or

$$-x(\tau) = \int_{\tau}^{\tau+T_0} F(\tau - t) Bw_\tau(t)^\circ dt \tag{9.11}$$

Let us choose the control $v_0(t) = w_\tau(t)^\circ \nu(\tau) / \zeta^\circ$. It satisfies the condition

$$\left[\int_{\tau}^{\tau+T_0} \|v_0(t)\|^2 dt\right]^{1/2} = \nu(\tau) \tag{9.12}$$

Consequently, from the meaning of the quantity T_0, we must find a control $u_0(t)$ which satisfies the condition

$$\left[\int_{\tau}^{\tau+T_0} \|u_0(t)\|^2 dt\right]^{1/2} \leqslant \mu(\tau) \tag{9.13}$$

and guarantees the encounter of the motions $y(t)$ and $z(t)$ at the instant $t = \tau + T_0$. Therefore the controls $u_0(t)$ and $v_0(t)$ must satisfy an equality analogous to the equality (9.11)

$$-x(\tau) = \int_{\tau}^{\tau+T_0} F(\tau - t) B[u_0(t) - v_0(t)] dt \tag{9.14}$$

Taking the values $v_0(t)$ and the equality (9.11) we get from (9.14) Equation

$$-\frac{\zeta^\circ + \nu(\tau)}{\zeta^\circ} x(\tau) = \int_{\tau}^{\tau+T_0} F(\tau - t) Bu_0(t) dt \tag{9.15}$$

which means that the control $w = u_0(t)$ solves the problem of changing the system (2.2) from the state $x = ([\zeta^\circ + \nu(\tau)] / \zeta^\circ) x(\tau)$ at the instant $t = \tau$ to the state $x(\tau + T_0) = 0$. The smallest norm $\rho^\circ_\tau[w]$ of the control $w(t)$, which solves such a problem according to (9.7) and (9.9) is the following:

$$\rho_{\tau,\tau+T_0}[w] = \zeta^\circ \left(\frac{\zeta^\circ + \nu(\tau)}{\zeta^\circ}\right) = \zeta^\circ + \nu(\tau) > \mu(\tau) \tag{9.16}$$

But the inequality (9.16) contradicts the assumption that $\rho_\tau[u_0] \leqslant \mu(\tau)$. This contradiction eliminates the inequality (9.9), and this, according to the previous reasoning, shows the equivalence of the quantities T^0 and T_0.

We shall show now that the rule (2.4) for choosing the controls $u^0(\tau)$ and $v^0(\tau)$ represents the aiming of the motions $y(t)$ and $z(t)$ at the point $\xi^0(\tau)$ at the instant $t = \tau + T^0 = \tau + T_0$.

In fact, the control $u(t) = w_\tau^\circ(t)\mu(\tau):(\mu(\tau) - \nu(\tau))$ brings the motion $y(t)$ to the boundary of the domain $H^{(1)}[y(\tau), \mu(\tau), \tau + T^\circ]$, at the instant $t = \tau + T^0$, since the control $w_\tau^\circ(t)$ is optimum for the problem (2.1) to (2.3).

Similarly, the control $v(t) = w_\tau^\circ(t)\nu(\tau):(\mu(\tau) - \nu(\tau))$ brings the motion $z(t)$ to the boundary of the domain $H^{(2)}[z(\tau), \nu(\tau), \tau + T^\circ]$ at the instant $t = \tau + T^0$.

Furthermore, those controls realize the encounter of the motions $y(t)$ and $z(t)$ at the time $t = \tau + T^0$. Consequently, the controls

$$u = u^\circ = \frac{w_\tau^\circ(\tau)\mu(\tau)}{\mu(\tau) - \nu(\tau)}, \qquad v = v^\circ = \frac{w_\tau^\circ(\tau)\nu(\tau)}{\mu(\tau) - \nu(\tau)} \tag{9.17}$$

aim the motions $y(t)$ and $z(t)$ at the single point $\xi^0(\tau)$ where the boundaries of the domains $H^{(1)}$ and $H^{(2)}$, intersect. These boundaries appear as similar figures, dimensions of which are in a ratio μ/ν. This proves our statement on the coincidence of the rules (2.4) and the rule related to the aiming at the point $\xi^0(\tau)$ of [1]. Taking into account the results formulated in the previous sections, we come to the following conclusion.

If for $t = \tau$ there were quantities $y(\tau)$, $z(\tau)$ and $\mu(\tau) > \nu(\tau)$, for which there exists a finite instant $t = \tau + T_0$ at which the process $y(t)$ (1.1) takes over the process $z(t)$ (1.1), and if for $t \geq \tau$ the control $u(t)$ is always chosen from the conditions of aiming the motion $y(t)$ at the point $\xi^0(t)$, then, in the pursuit process, the system, for all times before the encounter, is closed by a differential feedback loop, and the encounter occurs not later than for $t = \tau + T_0$.

Similarly, if $v(t)$ aims the motion $z(t)$ at the point $\xi^0(t)$, then the encounter occurs for $t = \tau + T_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$.

If, however, the control $u(t)$ does not follow the law which aims at the point $\xi^0(t)$, then the choice of the control $u(t)$ from the conditions aiming the motion $z(t)$ at the point $\xi^0(t)$ at all times for $t \geq \tau$, does not guarantee a time less than $T_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$, elapsing before the encounter, even if we limit ourselves to such controls $u(t)$, which for $t \geq \tau$, will keep finite values of $T_0[y(t), z(t), \mu(t), \nu(t)]$.

However, the combined control $v(t)$, which for $T_0[y(t), z(t), \mu(t), \nu(t)] < \theta$ and $\mu(t) - \nu(t) > \varepsilon_1 > 0$ is chosen from the conditions of aiming at the point $\xi^0(t)$, and for $T_0[y(t), z(t), \mu(t), v(t)] \geqslant \theta$ is chosen equal to $u(t - \eta)$ for sufficiently small $\eta > 0$, guarantees an encounter not earlier than at the moment $t = \tau + T_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau)] - \vartheta_*$, where ϑ_* is an arbitrary positive number chosen beforehand (see page 276).

Finally, let us describe a computational scheme, on the basis of which are constructed the controls u^0 or v^0 determined by the rule (2.4). This scheme follows from the rule [2], which describes the construction of an optimum control $w^\circ_\tau(t)$, which solves the problem (2.2),(2.3) of the limit time-optmum response. In agreement with this rule we must choose a functional space $\{h(t)\}_{[\tau,\vartheta]}$ $(\tau \leqslant t \leqslant \vartheta)$ with the norm $\varkappa_{\tau,\vartheta}[h(t)]$, for which the quantity $\rho_{\tau,\vartheta}[w(t)]$, which bounds the control resources, represents the norm of the linear functional

$$\varphi_w[h] = \int_\tau^\vartheta w^*(t)\, h(t)\, dt \tag{9.18}$$

determined on the functions $h(t)$. Then, we must solve the problem of the conditional minimum

$$\gamma(\vartheta - \tau) = \min \varkappa_{\tau,\vartheta}\Big[\sum_{i=1}^n l_i h^{(i)}(t)\Big] \quad \text{for } x^*(\tau)\, l = 1 \tag{9.19}$$

$$h_k^{(i)}(t) = \sum_{j=1}^n f_{ij}^{-1}(t-\tau)\, b_{jk} \qquad (\{f_{ij}^{-1}\} = F^{-1})$$

The number ϑ°, for which

$$\gamma(\vartheta^\circ - \tau) = [\mu(\tau) - \nu(\tau)]^{-1} = \zeta(\tau)^{-1} \tag{9.20}$$

determines the quantity $T^\circ[x(\tau),\ \zeta(\tau)] = \vartheta^\circ - \tau$.

The control $w^\circ_\tau(t)$ is determined from the maximum conditions

$$\int_\tau^{\vartheta^\circ} w_\tau^\circ(t)^*\, h^\circ(t)\, dt = \max_w \quad \text{for } \rho_{\tau,\vartheta^\circ}[w(t)] = \zeta(\tau) \tag{9.21}$$

After the determination of $w^\circ_\tau(\tau)$ the values of $u^\circ(\tau)$ and $v^\circ(\tau)$ are determined according to the relations (2.4).

Thus, if one of the partners stays with the rules (2.4), the procedure for the computation of the control $u^\circ(t)$ (or $v^\circ(t)$) at actual instants $t \geq \tau$ results in the continuous correction of the quantities $\vartheta^\circ(t)$ and of the functions $w^\circ_t(t)$ in agreement with Equation (9.20) and relation (9.21).

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